

A proof for Ando's theorem on norm-coherent coordinates via the Coleman norm operator

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Abstract

Ando established an algebraic criterion for when a complex orientation for a Morava E-theory is an H_∞ map. The criterion relates such an orientation to a specific property of the formal group associated to the E-theory, namely, a norm coherence condition on its coordinate. On the other hand, Coleman constructed a norm operator for interpolating division values in local fields, which depends on a Lubin–Tate formal group law. These formal group laws are important tools in explicit local class field theory.

In this article, we give a conceptual proof for Ando's theorem using the Coleman norm operator via the bridge of formal group laws between topology and arithmetic.

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1 Introduction

The purpose of this article is to supply a number-theoretic, conceptual proof for an old, topological theorem of Matthew Ando's from his study of structured ring spectra and their multiplicative operations. We follow a suggestion by Charles Rezk and Yifei Zhu (cf. [Zhu20, Remark 1.3]).

- We identify Ando's algebraic criterion for H_∞ complex orientations of Morava E-theories as a coherence condition for certain division values in local fields with respect to a Coleman norm operator.
- We then coherently construct these division values by an infinite iteration of the operator.

In a concrete way, the ingredients going into the proof hint towards a higher-algebraic theory for class fields (cf., e.g., [BSY22]).

Throughout, k denotes a perfect field of characteristic $p > 0$, R denotes a complete local ring with maximal ideal $\mathfrak{m}$ and whose residue field $R/\mathfrak{m}$ contains k . Given a formal group law $F \in R[[X, Y]]$, we will write $X +_F Y := F(X, Y)$.

Morava E-theories E_n are complex oriented cohomology theories in stable homotopy theory, whose coefficient ring $(E_n)_*$ in degree 0 classifies deformations of a 1-dimensional formal group law G of height n over k to R . Each E-theory is represented by an E_∞ -spectrum [GH04, Corollary 7.6] and carries power operations associated to this multiplicative structure. Let MU be the complex cobordism theory. It is well-known that MU also admits power operations ([tD68] and [May77, §IV.2]). A complex orientation on E_n is a map $\text{MU} \rightarrow E_n$ of homotopy commutative ring spectra. Upon taking the connective cover $\text{MU}\langle 0 \rangle$, this is equivalent to a coordinate (i.e., local uniformizer) of the universal deformation formal group associated with E_n , which in turn determines a formal group law [Zhu20, Section 2]. In the case that G is a Honda formal group law over $\mathbb{F}_p$, Ando gave a criterion for when power operations on MU and E_n are compatible along the map $\text{MU} \rightarrow E_n$, i.e., when this map is H_∞ [And95, Theorem 5]. The criterion is formulated in terms of the formal group law F of the universal deformation associated to the coordinate.

Theorem 1.1 ([And95, Theorem 4]). *Suppose $k = \mathbb{F}_p$ and G is the Honda formal group law of height n over k , so that $[p]_G(T) = T^{p^n}$, where $[p]_G$ is the p -series of G . Then in each $\star$ -isomorphism class of lifts of G to $\pi_0(E_n) \cong W(k)[[u_1, \dots, u_{n-1}]]$, there is a unique lift F such*

that

$$[p]_F(T) = \prod_{\lambda} (T +_F \lambda)$$

where the product runs over all roots λ of $[p]_F$.

Such a coordinate is said to be **norm-coherent**, in that the right-hand side has the form of a norm map. The condition says that the canonical lift of Frobenius, which is just the map of multiplication by p corresponding to the p -series in this case, coincides with the norm map. For a detailed discussion about the condition, see Section 4 and [Zhu20, Section 6].

On the number-theoretic side, an important tool invented in Lubin and Tate's explicit construction of the local Artin map in local class field theory is the family of Lubin–Tate formal group laws. Suppose a prime number p is a uniformizer of a local field K , i.e., K is an unramified extension of $\mathbb{Q}_p$. Then a Lubin–Tate formal group law reduces to a Honda formal group law over the residue field. In 1979, Coleman proved an interpolation theorem on division values in local fields, with applications to p -adic L -functions and modular units [Col79, Theorem A]. For that, he constructed a norm operator $\mathcal{N}_F$ depending on a Lubin–Tate formal group law F such that

$$\mathcal{N}_F(g) \circ [p]_F(T) = \prod_{\{\lambda: [p]_F(\lambda)=0\}} g(T +_F \lambda)$$

Rezk conjectured that the Coleman norm and Ando's algebraic criteria were closely related. Here, we prove Theorem 1.1 via the norm operators $\mathcal{N}_F$.

In more detail, the original definition of a norm operator restricts to the special case when R is a complete discrete valuation ring with uniformizer p . We shall first give a description for Coleman's norm operators and prove Ando's theorem in this special case. Our proof will only depend on several properties of the norm operators and do not require G to be a Honda formal group law. In view of this, we will generalize the definition of a norm operator to complete local domains in which $p \neq 0$, and show that the generalized norm operator satisfies the desired properties. In particular, $\pi_0(E_n)$ is a complete local domain with $p \neq 0$. The main result of this article is the following.

Theorem 1.2. *Suppose G is a formal group law of finite height over k and R is a complete local domain with $p \neq 0$ whose residue field contains k . Then in each $\star$ -isomorphism class of lifts of G to R , there is a unique lift whose corresponding coordinate is norm-coherent.*

A more precise and explicit formulation of Theorem 1.2 will be given as Theorem 4.8.

Remark 1.3. *Zhu generalized the above theorem to apply to arbitrary complete local rings R following the original proof by Ando [Zhu20, Theorem 1.2]. It would be interesting to have an alternative approach from number-theoretic constructions in this generality.*

2 Coleman norm operators from explicit local class field theory after Lubin and Tate

In this section, suppose that $k = \mathbb{F}_q$ with $q = p^n$. Suppose K is a local field with integer ring $\mathcal{O}_K$, maximal ideal $\mathfrak{m}$, and residue field k . Pick a uniformizer π of $\mathcal{O}_K$ and let

$$\mathcal{F}_\pi := \{\alpha(T) \in \mathcal{O}_K[[T]] : \alpha(T) \equiv \pi T \pmod{T^2}, \alpha(T) \equiv T^q \pmod{\pi}\} \quad (1)$$

In explicit local class field theory, we have the following.

Proposition 2.1. *For any $\alpha \in \mathcal{F}_\pi$, there is a unique formal group law F_α over $\mathcal{O}_K$ such that $\alpha \in \text{End}(F_\alpha)$.*

Proof. See [Mil20, I, 2.12]. □

Proposition 2.2. (a) *For any $\alpha, \beta \in \mathcal{F}_\pi$ and $a \in \mathcal{O}_K$, there is a unique element $[a]_{\beta, \alpha}(T)$ in $T\mathcal{O}_K[[T]]$ such that $[a]_{\beta, \alpha}(T) \equiv aT \pmod{T^2}$ and $[a]_{\beta, \alpha} \in \text{Hom}(F_\alpha, F_\beta)$.*

(b) *Moreover, the map $a \mapsto [a]_{\alpha, \alpha}(T)$ gives an isomorphism $\mathcal{O}_K \rightarrow \text{End}(F_\alpha)$. In particular, $\alpha(T) = [\pi]_{\alpha, \alpha}(T)$.*

Proof. See [Mil20, I, 2.14 and 2.17]. □

The formal group laws F_α characterized by Proposition 2.1 are the **Lubin–Tate formal group laws**. Let $\alpha \in \mathcal{F}_\pi$ and $\alpha = \tilde{\alpha}v$ for some unit $v \in \mathcal{O}_K[[T]]$ and some polynomial $\tilde{\alpha} \in \mathcal{O}_K[T]$ by the Weierstrass preparation theorem. We then define a finite set

$$\Lambda_{\alpha, 1} := \{\text{roots of } \tilde{\alpha} \text{ in a fixed algebraic closure of } K\} \quad (2)$$

Suppose $\mathcal{O}_K((T))$ is the ring of formal Laurent series with coefficients in $\mathcal{O}_K$. We assign the “compact–open” topology to $\mathcal{O}_K((T))$, i.e., a sequence $\{g_n\}$ converges to g if and only if for any compact subset A not containing 0 in $\mathfrak{m}$, and for each $\epsilon > 0$, there exists a positive integer

$N = N(A, \epsilon)$ such that $|g_n(a) - g(a)| < \epsilon$ for all $a \in A$ and $n \geq N$. Given a Lubin–Tate formal group law F_α , the Coleman norm operator is characterized by the following.

Theorem 2.3 ([Col79, Theorem 11 and Corollary 12]). *As notations above, there exists a unique function $\mathcal{N}_{F_\alpha} : \mathcal{O}_K((T)) \rightarrow \mathcal{O}_K((T))$ satisfying*

$$\mathcal{N}_{F_\alpha}(g) \circ [p]_{F_\alpha}(T) = \prod_{\lambda \in \Lambda_{\alpha,1}} g(T +_{F_\alpha} \lambda)$$

for every $g \in \mathcal{O}_K((T))$. Moreover, $\mathcal{N}_{F_\alpha}$ is continuous and multiplicative.

The norm operator has the following properties.

Lemma 2.4. *Let $i \geq 1$, $g \in 1 + \mathfrak{m}^i[[T]]$ and h be a unit in $\mathcal{O}_K((T))$. Then*

- (a) $\mathcal{N}_{F_\alpha}(g) \in 1 + \mathfrak{m}^{i+1}[[T]]$ and
- (b) $\mathcal{N}_{F_\alpha}^i(h) / \mathcal{N}_{F_\alpha}^{i-1}(h) \in 1 + \mathfrak{m}^i[[T]]$, where $\mathcal{N}_{F_\alpha}^i$ denotes i iterations of applying the norm operator $\mathcal{N}_{F_\alpha}$.

Proof. See [Col79, Lemma 13]. Here, part (b) looks different from [Col79, Lemma 13(b)], $\mathcal{N}_{F_\alpha}^i(h) / \phi \mathcal{N}_{F_\alpha}^{i-1}(h) \in 1 + \pi^i \mathcal{O}_K[[T]]$, where $\phi \in \text{Gal}(H/K)$ is the Frobenius map for a complete unramified extension H/K and π is the corresponding uniformizer. For our applications, we need only consider K itself with ϕ the identity. $\square$

As a consequence of Lemma 2.4(b), the sequence $\{\mathcal{N}_{F_\alpha}^i(h)\}$ converges in $\mathcal{O}_K((T))$. Let $\mathcal{N}_{F_\alpha}^\infty(h) := \lim_{i \rightarrow \infty} \mathcal{N}_{F_\alpha}^i(h)$. In particular, by Lemma 2.4(a), we have that

$$\mathcal{N}_{F_\alpha}^\infty(1 + \mathfrak{m}[[T]]) = 1 \tag{3}$$

Since $\mathcal{N}_{F_\alpha}$ is continuous,

$$\mathcal{N}_{F_\alpha}(\mathcal{N}_{F_\alpha}^\infty(h)) = \mathcal{N}_{F_\alpha}\left(\lim_{i \rightarrow \infty} \mathcal{N}_{F_\alpha}^i(h)\right) = \lim_{i \rightarrow \infty} \mathcal{N}_{F_\alpha}(\mathcal{N}_{F_\alpha}^i(h)) = \mathcal{N}_{F_\alpha}^\infty(h)$$

Moreover, the operator $\mathcal{N}_{F_\alpha}^\infty$ is multiplicative since $\mathcal{N}_{F_\alpha}$ is.

3 Proof of Ando's theorem in a special case

We will first prove Theorem 1.1 (and 1.2) in a special case using Coleman norm operators.

In this section, suppose that K is an unramified extension of $\mathbb{Q}_p$ of degree n and G is the Honda formal group law over $k \cong \mathbb{F}_{p^n}$ of height n . Here, p is a uniformizer of K .

In (1), choose the uniformizer $\pi = p$. Given any $\alpha \in \mathcal{F}_\pi$, let F_α be the associated Lubin–Tate formal group law so that $[p]_{F_\alpha}(T) = [\pi]_{\alpha, \alpha}(T) = \alpha(T)$ by Proposition 2.2(b). Thus, F_α is a lift of G to $\mathcal{O}_K$. Conversely, every lift of G to $\mathcal{O}_K$ has p -series in $\mathcal{F}_\pi$, so by the uniqueness in Proposition 2.1 it is a Lubin–Tate formal group law.

Definition 3.1 ($\star$ -isomorphisms). Two formal group laws F, F' over $\mathcal{O}_K$ are said to be $\star$ -*isomorphic* if there is an isomorphism $u: F \rightarrow F'$ such that u restricts to the identity series to k . A more general definition of $\star$ -isomorphism will be given in Section 4.

Theorem 3.2. *With notations as above and in (2), in each $\star$ -isomorphism class of lifts of G to $\mathcal{O}_K$, there is a unique formal group law $F = F_\alpha$, necessarily a Lubin–Tate formal group law for some $\alpha \in \mathcal{F}_\pi$, satisfying*

$$[p]_{F_\alpha}(T) = \prod_{\lambda \in \Lambda_{\alpha,1}} (T +_{F_\alpha} \lambda) \quad (4)$$

To prove this special case of Ando’s theorem, we proceed as follows. In terms of the norm operator, we see that a Lubin–Tate formal group law satisfies (4) if and only if

$$[p]_{F_\alpha}(T) = \prod_{\lambda \in \Lambda_{\alpha,1}} (T +_{F_\alpha} \lambda) = (\mathcal{N}_{F_\alpha}(T) \circ [p]_{F_\alpha})(T) \quad (5)$$

Since p is invertible in K , $[p]_{F_\alpha}$ has a composition inverse in $K[[T]]$. We can thus cancel the term from both sides above, so that condition (4) is equivalent to

$$\mathcal{N}_{F_\alpha}(T) = T \quad (6)$$

Begin with any lift F_α of G to $\mathcal{O}_K$. Let $u \in T + \pi T \mathcal{O}_K[[T]] = T + T\mathfrak{m}[[T]]$. Then there is an element $\alpha_u \in \mathcal{F}_\pi$ such that $u \circ F_\alpha \circ u^{-1} = F_{\alpha_u}$. Indeed, since $\alpha = [p]_{F_\alpha}$ and $\alpha_u = [p]_{F_{\alpha_u}}$, we have $\alpha_u = u \circ \alpha \circ u^{-1}$. Clearly F_α and F_{α_u} are $\star$ -isomorphic. In order to show that there is a unique $u \in T + T\mathfrak{m}[[T]]$ such that F_{α_u} satisfies (4), we are reduced to show that there is a unique $u \in T + T\mathfrak{m}[[T]]$ such that

$$\mathcal{N}_{F_{\alpha_u}}(T) = T \quad (7)$$

Note that u induces a bijection from $\Lambda_{\alpha,1}$ to $\Lambda_{\alpha u,1}$. By definition,

$$(\mathcal{N}_{F_{\alpha u}}(T) \circ [p]_{F_{\alpha u}})(T) = \prod_{\lambda \in \Lambda_{\alpha u,1}} (T +_{F_{\alpha u}} \lambda)$$

We rewrite this identity as follows:

$$\begin{aligned} (\mathcal{N}_{F_{\alpha u}}(T) \circ u \circ [p]_{F_{\alpha}} \circ u^{-1})(T) &= \prod_{\lambda \in \Lambda_{\alpha,1}} (T +_{F_{\alpha u}} u(\lambda)) \\ &= \prod_{\lambda \in \Lambda_{\alpha,1}} F_{\alpha u} \left(u(u^{-1}(T)), u(\lambda) \right) \\ &= \prod_{\lambda \in \Lambda_{\alpha,1}} u \circ F_{\alpha} (u^{-1}(T), \lambda) \\ &= \prod_{\lambda \in \Lambda_{\alpha,1}} u \circ (u^{-1}(T) +_{F_{\alpha}} \lambda) \\ &= (\mathcal{N}_{F_{\alpha}}(u) \circ [p]_{F_{\alpha}})(u^{-1}(T)) \end{aligned}$$

By canceling $[p]_{F_{\alpha}} \circ u^{-1}$ from both sides, we obtain $(\mathcal{N}_{F_{\alpha u}}(T) \circ u)(T) = \mathcal{N}_{F_{\alpha}}(u)(T)$. Therefore, (7) is equivalent to

$$\mathcal{N}_{F_{\alpha}}(u) = u$$

Consequently, it remains to show the following.

Proposition 3.3. *Given any $\alpha \in \mathcal{F}_{\pi}$, there is a unique $u \in T + T\mathfrak{m}[[T]]$, such that $\mathcal{N}_{F_{\alpha}}(u) = u$.*

Proof. Existence: By Lemma 2.4(b), let $h_i := \mathcal{N}_{F_{\alpha}}^i(T) / \mathcal{N}_{F_{\alpha}}^{i-1}(T) \in 1 + \mathfrak{m}^i[[T]]$. Then we have $\mathcal{N}_{F_{\alpha}}^{\infty}(T) = Th_1h_2\cdots$. It is easy to see that $h_1h_2\cdots \in 1 + \mathfrak{m}[[T]]$, so $\mathcal{N}_{F_{\alpha}}^{\infty}(T) \in T + T\mathfrak{m}[[T]]$. Therefore, $u = \mathcal{N}_{F_{\alpha}}^{\infty}(T)$ satisfies the condition.

Uniqueness: If $\mathcal{N}_{F_{\alpha}}(u) = u$, then $\mathcal{N}_{F_{\alpha}}^i(u) = u$ for each i . Thus, $\mathcal{N}_{F_{\alpha}}^{\infty}(u) = u$ after taking the limit. Since $u \in T + T\mathfrak{m}[[T]]$, there is $\tilde{u} \in 1 + \mathfrak{m}[[T]]$ such that $u = T\tilde{u}$. Then, in view of (3),

$$u = \mathcal{N}_{F_{\alpha}}^{\infty}(u) = \mathcal{N}_{F_{\alpha}}^{\infty}(T) \mathcal{N}_{F_{\alpha}}^{\infty}(\tilde{u}) = \mathcal{N}_{F_{\alpha}}^{\infty}(T)$$

□

Remark 3.4. *We can interpret the equality $\mathcal{N}_{F_{\alpha}}(u) = u$ as a condition of norm coherence in the context of [Col79]. To be precise, suppose $\Lambda_{\alpha,n}$ is the set of roots of $[p^n]_{F_{\alpha}}$ in the fixed algebraic closure of K . Let $K_{\pi,n} := K(\Lambda_{\alpha,n})$ and $N_{n+1,n} := N_{K_{\pi,n+1}/K_{\pi,n}}$ be the norm map.*

It can be shown that $\Lambda_{\alpha,n} \cong \mathcal{O}_K/\mathfrak{m}^n$ [Mil20, I, 3.4]. Suppose v_n is a generator of $\Lambda_{\alpha,n}$ as an $\mathcal{O}_K$ -module for each n such that $[p]_{F_\alpha}(v_{n+1}) = v_n$. We then have

$$\mathcal{N}_{F_\alpha}(u)(v_n) = N_{n+1,n}(u(v_{n+1}))$$

by [Col79, Corollary 12(ii)]. Thus, $\mathcal{N}_{F_\alpha}(u) = u$ is equivalent to saying that

$$u(v_n) = N_{n+1,n}(u(v_{n+1}))$$

i.e., u transforms the sequence $\{v_n\}$ to a sequence compatible with the norm maps.

Remark 3.5. Let $\mathcal{M}_{\infty,\alpha} = \{g \in \mathcal{O}_K((T))^* : \mathcal{N}_{F_\alpha}(g) = g\}$ be the subset in $\mathcal{O}_K((T))^*$ consisting of norm-coherent series in the sense above. Then the uniqueness of u follows from an exact sequence of groups.

$$1 \rightarrow 1 + \mathfrak{m}[[T]] \rightarrow \mathcal{O}_K((T))^\times \xrightarrow{\mathcal{N}_{F_\alpha}^\infty} \mathcal{M}_{\infty,\alpha} \rightarrow 1$$

as in [Col79, Proposition 14].

4 Norm coherence condition

In a more general case of Theorem 3.2, the left-hand side of (4) should not simply be $[p]_{F_\alpha}(T)$. Indeed, it should be a canonical lift of the relative Frobenius map. To define this lift, we have to first recall some notions in formal groups and deformations of formal group laws.

Suppose $\mathcal{F}$ is a formal group over R and $\mathcal{G}$ is a formal group over k of finite height n . Let F and G be the respective formal group laws associated to chosen coordinates of these formal groups. Let X be a chosen coordinate on $\mathcal{F}$.

Definition 4.1 (Quotients of formal groups). Suppose $\mathcal{D}$ is a subgroup of $\mathcal{F}$ of degree p^r defined over a complete local ring S containing R . Then we define the **quotient group** $\mathcal{F}/\mathcal{D}$ over S as follows. Let $m: \mathcal{F} \times \mathcal{D} \rightarrow \mathcal{F}$ be the multiplication map and $pr_1: \mathcal{F} \times \mathcal{D} \rightarrow \mathcal{F}$ be the projection onto the first factor. The coordinate ring of $\mathcal{F}/\mathcal{D}$ is defined by the equalizer

$$\mathcal{O}_{\mathcal{F}/\mathcal{D}} \xrightarrow{f_{\mathcal{D}}^*} \mathcal{O}_{\mathcal{F}} \xrightarrow[\text{pr}_1^*]{m^*} \mathcal{O}_{\mathcal{F} \times \mathcal{D}}$$

It can be shown that $\mathcal{F}/\mathcal{D}$ is a formal group over S . In addition, viewing $\mathcal{O}_{\mathcal{F}}$ as an $\mathcal{O}_{\mathcal{F}/\mathcal{D}}$ -module through $f_{\mathcal{D}}^*$, we obtain $X_{\mathcal{D}} = \text{Norm}_{f_{\mathcal{D}}^*}(X)$ as a coordinate of $\mathcal{F}/\mathcal{D}$ [Str97, Theorem 19].

If $\mathcal{D}(S)$ contains exactly p^r elements, then

$$f_{\mathcal{D}}^*(X_{\mathcal{D}}) = \prod_{P \in \mathcal{D}(S)} (X +_F X(P)) \quad (8)$$

by direct calculation. Thus, reducing (8) to the residue field of R we have

$$f_{\mathcal{D}}^*(X_{\mathcal{D}}) \equiv X^{p^r} \pmod{\mathfrak{m}} \quad (9)$$

Notation. For simplicity, from now on, we will not distinguish an isogeny between formal groups and the power series to which it corresponds as a map between coordinate rings. For instance, we will simply write

$$f_{\mathcal{D}}(T) = \prod_{P \in \mathcal{D}(S)} (T +_F X(P))$$

as an isogeny between formal group laws.

The following definition generalizes Definition 3.1.

Definition 4.2 (Deformations of formal group laws and $\star$ -isomorphisms). Let $\pi: R \rightarrow R/\mathfrak{m}$ be the natural projection. A **deformation of G to R** is a triple (F, i, η) , where F is a formal group law over R , $i: k \rightarrow R/\mathfrak{m}$ is an inclusion and $\eta: \pi^*(F) \rightarrow i^*(G)$ is an isomorphism. Here π and i act on each coefficient.

Suppose (F, i, η) and (F', i', η') are two deformations of G to R such that $i = i'$. Then we say (F, i, η) and (F', i', η') are **$\star$ -isomorphic** if there is an isomorphism $\psi: F \rightarrow F'$ of formal group laws such that $\eta' \circ \pi^*(\psi) = \eta$.

Furthermore, if $\eta = \eta'$ and $\pi^*(\psi) = \text{id}$, we call $\psi: F \rightarrow F'$ a **$\star$ -isomorphism** as well.

Definition 4.3 (Deformations of Frobenius). Suppose Φ is the relative Frobenius map on G over k and σ is the absolute Frobenius map over k . Suppose (F, i, η) and (F', i', η') are two deformations of G to R . An isogeny $\psi: F \rightarrow F'$ of degree p^r is a **deformation of Frobenius** if $i' = i \circ \sigma^r$, and $\eta' \circ \pi^*(\psi) = i^*(\Phi^r) \circ \eta$, i.e., the following diagram commutes.

$$\begin{array}{ccccc}
\mathcal{F} & \xleftarrow{\quad} & \pi^*(\mathcal{F}) & \xrightarrow{\quad \eta \quad} & i^*(\mathcal{G}) \\
\downarrow \psi & & \downarrow \pi^*(\psi) & & \downarrow i^*(\Phi^r) \\
\mathcal{F}' & \xleftarrow{\quad} & \pi^*(\mathcal{F}') & \xrightarrow[\eta']{\quad} & i^*(\mathcal{G}^{(p^r)}) \\
& & & & \parallel \\
& & & & i'^*(\mathcal{G})
\end{array}$$

Two deformations of Frobenius $(F_1, i_1, \eta_1) \rightarrow (F'_1, i'_1, \eta'_1)$ and $(F_2, i_2, \eta_2) \rightarrow (F'_2, i'_2, \eta'_2)$ are **isomorphic** if $(F_1, i_1, \eta_1), (F_2, i_2, \eta_2)$ are $\star$ -isomorphic and $(F'_1, i'_1, \eta'_1), (F'_2, i'_2, \eta'_2)$ are $\star$ -isomorphic.

The following theorem classifies deformations of Frobenius.

Theorem 4.4 (cf. [Str97, Theorem 42]). *There is a universal deformation $(F_{\text{univ}}, \text{id}, \text{id})$ of G to the Lubin–Tate ring $W(k)[[u_1, \dots, u_{n-1}]]$, in the following sense. For each $r \geq 0$, there is a complete local ring A^r such that*

$$\{\text{deformations } (F, i, \eta) \rightarrow (F', i', \eta') \text{ of } \Phi^r \text{ to } T\} / \text{isomorphisms} \cong \text{Hom}(A^r, T)$$

Moreover, $A^0 = W(k)[[u_1, \dots, u_{n-1}]]$ and A^r is a bimodule over A^0 with structure maps $s^r, t^r: A^0 \rightarrow A^r$, which are local homomorphisms.

The isomorphism is given as follows: for any deformation $(F, i, \eta) \rightarrow (F', i', \eta')$ of Φ^r to T , there is a unique local homomorphism $\rho^r: A^r \rightarrow T$ such that $\rho^r \circ s^r$ and $\rho^r \circ t^r$ restrict to i and i' on the residue fields respectively and that there are unique $\star$ -isomorphisms $(F, i, \eta) \rightarrow (\rho^{r*} s^{r*} F_{\text{univ}}, i, \text{id})$ and $(F', i', \eta') \rightarrow (\rho^{r*} t^{r*} F_{\text{univ}}, i', \text{id})$.

Remark 4.5. According to [Zhu20, Remark 6.2], given any deformation (F, i, η) of G to R , there exists a unique deformation $(\tilde{F}, i, \text{id})$ given by [LT66, Theorem 3.1] such that the two deformations are $\star$ -isomorphic. Thus, we will assume $\eta = \text{id}$ in the following.

Suppose from now on that (F, i, id) is a deformation of G to R , where R is a complete local domain with $p \neq 0$.

Since $[p]_F \equiv [p]_G \pmod{\mathfrak{m}}$ and G has height $n < \infty$, not all coefficients of $[p]_F$ are in $\mathfrak{m}$. By the Weierstrass preparation theorem, there is a unit v in $R[[T]]$ and a monic polynomial β of degree p^n , such that $[p]_F = v \cdot \beta$. Note that roots of $[p]_F$ are the same as the roots of β . Let

$$\Lambda_{[p]_F, 1} := \{\text{roots of } [p]_F \text{ in a larger ring } S \text{ obtained from } R \text{ by adjoining roots of } \beta\} \quad (10)$$

Since $p \neq 0$ in R , 0 is a simple root of $[p]_F$. For any $\lambda \in \Lambda_{[p]_F, 1}$, $[p]_F(T -_F \lambda) = [p]_F(T)$. Therefore, λ is also a simple root of $[p]_F$. We conclude that the set $\Lambda_{[p]_F, 1}$ has exactly p^n elements.

Let $\mathcal{D} := \mathcal{F}[p]$ be the subgroup of $\mathcal{F}$ of p -torsions defined over S . Then the quotient isogeny $f_p: \mathcal{F} \rightarrow \mathcal{F}/\mathcal{F}[p]$ is given by

$$f_p(T) = \prod_{\lambda \in \Lambda_{[p]_F, 1}} (T +_F \lambda)$$

Note that $f_p(T)$ is invariant under the action of $\text{Aut}(S/R)$, so $f_p(T) \in R[[T]]$. As a consequence, $\mathcal{F}/\mathcal{F}[p]$ can be defined over R . Thus, f_p is a deformation of Frobenius between (F, i, id) and $(F/F[p], i \circ \sigma^n, \text{id})$ in view of (9). By Theorem 4.4, there exists a unique local homomorphism $\rho^n: A^n \rightarrow R$ with a unique $\star$ -isomorphism $g_p: F/F[p] \rightarrow \rho^{n*} t^{n*} F_{\text{univ}}$.

Definition 4.6. Define

$$l_p := g_p \circ f_p: F \rightarrow \rho^{n*} t^{n*} F_{\text{univ}}$$

Remark 4.7. According to [Zhu20, Remark 6.7], the isogeny l_p is an isogeny of formal group laws over R characterized by the following properties.

- (a) It is an isogeny from F to $\rho^{n*} t^{n*} F_{\text{univ}}$.
- (b) The kernel of l_p is the same as that of $[p]_F$.
- (c) The reduction of l_p to the residue field is the p^n -power relative Frobenius T^{p^n} , i.e., $l_p(T) \equiv T^{p^n} \pmod{\mathfrak{m}}$.

To proceed to the general case, we formulate the norm coherence condition of Theorem 1.2 precisely as follows.

Theorem 4.8. Suppose R is a complete local domain with $p \neq 0$ and residue field containing k . In each $\star$ -isomorphism class of deformations of G to R , there is a unique element (F, i, id) such that

$$l_p(T) = f_p(T) = \prod_{\lambda \in \Lambda_{[p]_F, 1}} (T +_F \lambda) \tag{11}$$

i.e., $g_p(T) = T$.

Remark 4.9. According to the proof for [Zhu20, Proposition 7.1], the coordinate given by Theorem 4.8 is norm-coherent with respect to any finite subgroup $D \subset \mathcal{F}$ in the sense of [Zhu20, Definition 6.21].

Remark 4.10. Back to the topological side, an orientation on the Morava E -theory E_n corresponds to a coordinate on the formal group $\mathrm{Spf}(\pi_0(E_n^{\mathbb{CP}^\infty_+}))$ [AHS01, Example 2.53], so it induces a deformation (F, i, η) of G to $\pi_0(E_n^{\mathbb{CP}^\infty_+})$. A change of orientation on E_n will induce a $\star$ -isomorphism between the induced deformations [Zhu20, Example 4.9]. One should be aware that the $\star$ -isomorphism here may change the η -component. Recall that we set $\eta = \mathrm{id}$ in Remark 4.5. Thus, Theorem 4.8 implies that there is a unique H_∞ -orientation $\mathrm{MU}\langle 0 \rangle \rightarrow E_n$ such that the induced deformation has $\eta = \mathrm{id}$ and satisfies (11).

In general, we say a deformation (F, i, η) (also the orientation inducing this deformation) is **norm-coherent** if the component $\tilde{F}$ in $(\tilde{F}, i, \mathrm{id})$ is norm-coherent in the sense of (11), where the latter deformation is given by Remark 4.5 (cf. [Zhu20, Definition 6.21]). Therefore, there is only a family of norm-coherent orientations on E_n , which are $\star$ -isomorphic to the deformation (F, i, id) given by Theorem 4.8.

5 Generalization of the norm operators

In this section, we aim to prove Theorem 4.8 (and hence Theorem 1.2) following the proof of the special case in Section 3. Observe that our earlier proof actually only requires $\mathcal{O}_K$ to be a complete local domain such that $[p]_{F_\alpha}$ is right-cancellative and the definition and properties of the Coleman norm operator over $\mathcal{O}_K$ in Section 2. Therefore, we need only show that l_p is right-cancellative and generalize Theorem 2.3 and Lemma 2.4 to the case of a complete local domain R such that $p \neq 0$ and residue field contains k substituting $[p]_F$ by l_p . The proof then applies mutatis mutandis.

Remark 5.1. In the general case, we do not require G to be a Honda formal group law and F to be a Lubin–Tate formal group law as in Section 3, since the property that $[p]_F \equiv T^{p^n} \pmod{\mathfrak{m}}$ has been replaced by Remark 4.7(c).

Lemma 5.2. The power series l_p is right-cancellative, i.e., if there are $g, h \in R[[T]]$ such that $g \circ l_p = h \circ l_p$, we have $g = h$.

Proof. We may assume $h = 0$, so that $g(l_p(T)) = 0$. Since R is complete with respect to $\mathfrak{m}$, we need only prove by induction on i that $g \equiv 0 \pmod{\mathfrak{m}^i}$ for each $i \geq 0$. The statement is vacuous for $i = 0$. Suppose we have proven that $g \equiv 0 \pmod{\mathfrak{m}^i}$. Since $l_p(T) \equiv T^{p^n} \pmod{\mathfrak{m}}$ by Remark 4.7(c), $g(T^{p^n}) \equiv 0 \pmod{\mathfrak{m}^{i+1}}$, and hence $g \equiv 0 \pmod{\mathfrak{m}^{i+1}}$. $\square$

The following proofs essentially follow those in [Col79].

Lemma 5.3 (cf. [Col79, Lemma 3]). *If $g \in R[[T]]$ and $g(T +_F \lambda) = g(T)$ for all $\lambda \in \Lambda_{[p]_F, 1}$, then there is a unique $h \in R[[T]]$ such that $h \circ l_p = g$.*

Proof. The uniqueness follows from Lemma 5.2.

For the existence, we inductively construct formal power series g_m for each $m \geq 0$. Let $g_0 = g$. Suppose that we have constructed $a_i \in R$ for $0 \leq i \leq m-1$ such that

$$g - \sum_{i=0}^{m-1} a_i l_p^i = l_p^m \cdot g_m$$

for some $g_m \in R[[T]]$. Note that by assumption $g(T +_F \lambda) = g(T)$ and by Remark 4.7(b) $l_p(T +_F \lambda) = l_p(T)$. We thus have $g_m(T +_F \lambda) = g_m(T)$. In particular, $(g_m - g_m(0))(\lambda) = 0$ for all $\lambda \in \Lambda_{[p]_F, 1}$. By the Euclidean algorithm for power series, there exists elements $g_{m+1} \in R[[T]]$ with $r_m \in R[[T]]$ such that $g_m - g_m(0) = l_p \cdot g_{m+1} + r_m$ and $\deg(r_m) < p^n$. Then r_m vanishes on $\Lambda_{[p]_F, 1}$. Since we have deduced in the paragraph after (10) that $\Lambda_{[p]_F, 1}$ has p^n elements, we have $r_m = 0$. Let $a_m = g_m(0)$. This finishes the inductive step. Thus, we obtain

$$g - \sum_{i=0}^{\infty} a_i l_p^i \in \bigcap_{i=0}^{\infty} l_p^i R[[T]] = 0$$

Then $h(T) := \sum_{i=0}^{\infty} a_i T^i$ is the desired element. □

Now we also endow $R[[T]]$ with the compact–open topology similar to that on $\mathcal{O}_K[[T]]$ as in Section 2. Here R has the $\mathfrak{m}$ -adic topology.

Theorem 5.4 (cf. Theorem 2.3). *There is a unique operator $\mathcal{N}_F: R[[T]] \rightarrow R[[T]]$ such that for any $g \in R[[T]]$,*

$$\mathcal{N}_F(g) \circ l_p(T) = \prod_{\lambda \in \Lambda_{[p]_F, 1}} g(T +_F \lambda)$$

Moreover, $\mathcal{N}_F$ is multiplicative and continuous.

Proof. Note that the right-hand side of the above identity satisfies the condition of $\Lambda_{[p]_F, 1}$ -invariance from Lemma 5.3. Thus, there is a unique $\mathcal{N}_F$ satisfying the formula.

Given any $g, h \in R[[T]]$,

$$\begin{aligned}
\mathcal{N}_F(gh) \circ l_p(T) &= \prod_{\lambda \in \Lambda_{[p]_F, 1}} gh(T +_F \lambda) \\
&= (\mathcal{N}_F(g) \circ l_p(T)) \cdot (\mathcal{N}_F(h) \circ l_p(T)) \\
&= (\mathcal{N}_F(g) \cdot \mathcal{N}_F(h)) \circ l_p(T)
\end{aligned}$$

By Lemma 5.2, we then obtain $\mathcal{N}_F(gh) = \mathcal{N}_F(g) \cdot \mathcal{N}_F(h)$.

Suppose $\{g_n\}$ converges to g . Then

$$\begin{aligned}
(\lim \mathcal{N}_F(g_n)) \circ l_p(T) &= \lim (\mathcal{N}_F(g_n) \circ l_p(T)) = \lim \prod_{\lambda \in \Lambda_{[p]_F, 1}} g_n(T +_F \lambda) \\
&= \prod_{\lambda \in \Lambda_{[p]_F, 1}} g(T +_F \lambda) = \mathcal{N}_F(g) \circ l_p(T)
\end{aligned}$$

Again, it follows from Lemma 5.2 that $\lim \mathcal{N}_F(g_n) = \mathcal{N}_F(g)$. $\square$

Note that in the proof in Section 3 we only took limits of $\mathcal{N}_{F_\alpha}$ applied to elements in $1 + \mathfrak{m}[[T]]$ and to T . Thus, it remains to show the following for $\mathcal{N}_F$.

Lemma 5.5 (cf. Lemma 2.4). *Let $g \in 1 + \mathfrak{m}^i[[T]]$ and $i \geq 1$. Then*

- (a) $\mathcal{N}_F(g) \in 1 + \mathfrak{m}^{i+1}[[T]]$ and
- (b) $\mathcal{N}_F^i(T) / \mathcal{N}_F^{i-1}(T) \in 1 + \mathfrak{m}^i[[T]]$.

Proof. (a) By definition, $\mathcal{N}_F(g) \circ l_p(T) = \prod_{\lambda \in \Lambda_{[p]_F, 1}} g(T +_F \lambda)$. Suppose $g(T) = 1 + \sum_{j=0}^{\infty} b_j T^j$, where $b_j \in \mathfrak{m}^i$. Since $i \geq 1$, terms divided by $b_{j_1} b_{j_2}$ for some j_1, j_2 must lie in $\mathfrak{m}^{i+1}$. Therefore, modulo $\mathfrak{m}^{i+1}$,

$$\begin{aligned}
\mathcal{N}_F(g) \circ l_p(T) &\equiv 1 + \sum_{\lambda \in \Lambda_{[p]_F, 1}} \sum_{j=0}^{\infty} b_j (T +_F \lambda)^j \\
&= 1 + \sum_{j=0}^{\infty} \sum_{\lambda \in \Lambda_{[p]_F, 1}} b_j (T +_F \lambda)^j \\
&= 1 + \sum_{j=0}^{\infty} b_j (p^n T^j + \sum_{k=0}^{\infty} p_k (\Lambda_{[p]_F, 1}) T^k)
\end{aligned}$$

where each $p_k(\Lambda_{[p]_F, 1})$ is a symmetric function on $\lambda \in \Lambda_{[p]_F, 1}$. Recall from Section 4 that β is a polynomial of degree p^n dividing $[p]_F$ from the Weierstrass preparation

theorem, and $\Lambda_{[p]_F,1}$ is the set of roots of β . Thus, $\beta(T) \equiv T^{p^n} \pmod{\mathfrak{m}}$. It follows that $p_k(\Lambda_{[p]_F,1}) \equiv 0 \pmod{\mathfrak{m}}$, since $p_k(\Lambda_{[p]_F,1})$ is a polynomial in the non-leading coefficients of β (i.e., those elementary symmetric functions). Therefore,

$$\mathcal{N}_F(g) \circ l_p(T) \equiv 1 \pmod{\mathfrak{m}^{i+1}}$$

Next we prove by induction on j that if $h \in R[[T]]$ and $h \circ l_p \in \mathfrak{m}^j[[T]]$, then $h \in \mathfrak{m}^j[[T]]$ (here $j \geq 0$). Setting $h = \mathcal{N}_F(g) - 1$ completes the proof of part (a). The case is vacuous when $j = 0$. Suppose $j \geq 1$ and the statement holds for $j - 1$. By the induction hypothesis, $h \in \mathfrak{m}^{j-1}[[T]]$. Since $l_p(T) \equiv T^{p^n} \pmod{\mathfrak{m}}$, we have

$$h \circ l_p(T) \equiv h(T^{p^n}) \pmod{\mathfrak{m}^j}$$

Since $h \circ l_p \in \mathfrak{m}^j[[T]]$, we must have $h \in \mathfrak{m}^j[[T]]$.

- (b) By part (a) and the multiplicativity of $\mathcal{N}_F$ from Theorem 5.4, we need only show the case when $i = 1$. Since $l_p(T) \equiv T^{p^n} \pmod{\mathfrak{m}}$,

$$\mathcal{N}_F(T)(T^{p^n}) \equiv \mathcal{N}_F(T) \circ l_p(T) = \prod_{\lambda \in \Lambda_{[p]_F,1}} (T +_F \lambda) \pmod{\mathfrak{m}}$$

By an argument with symmetric functions similar to that for part (a), we obtain that $\prod_{\lambda \in \Lambda_{[p]_F,1}} (T +_F \lambda) \equiv T^{p^n} \pmod{\mathfrak{m}}$. Hence, $\mathcal{N}_F(T) \equiv T \pmod{\mathfrak{m}}$, so $\mathcal{N}_F(T)/T \equiv 1 \pmod{T^{-1}\mathfrak{m}[[T]]}$. It remains to show that T divides $\mathcal{N}_F(T)$ in $R[[T]]$, or $\mathcal{N}_F(T)(0) = 0$. Since $0 \in \Lambda_{[p]_F,1}$,

$$\mathcal{N}_F(T)(0) = \mathcal{N}_F(T) \circ l_p(0) = \prod_{\lambda \in \Lambda_{[p]_F,1}} \lambda = 0$$

□

Remark 5.6. Comparing (5) from the proof in Section 3, we see that f_p as defined in Section 4 now factors as $\mathcal{N}_F(T) \circ l_p$. Recall that $l_p = g_p \circ f_p$ by definition. By Lemma 5.2, l_p is right-cancellative with respect to composition, and hence so does f_p . Therefore, $\mathcal{N}_F(T)$ turns out to be the inverse to g_p , i.e., it is a (necessarily unique) $\star$ -isomorphism $\rho^{n*}t^{n*}F_{\text{univ}} \rightarrow F/F[p]$. This agrees with the construction of the coordinate associated with $F/F[p]$ in Definition 4.1.

In view of this, the proof in Section 3 is actually similar to the one for [And95, Theorem 2.6.4]. In [And95, Theorem 2.6.4], Ando constructed a series of $\star$ -isomorphisms to modify the coordinates such that $g_p(T)$ can approximate T inductively modulo $\mathfrak{m}^r$. On the other hand, our proof constructed a $\star$ -isomorphism $u = \mathcal{N}_F^\infty(T)$ directly so that $g_p^{-1}(T) = \mathcal{N}_F(T)(T) = T$ after a change of coordinate by u . Indeed, properties of this limit of operators (i.e., Lemma 2.4 and Lemma 5.5) are deduced inductively on r modulo $\mathfrak{m}^r$ similar to the one in Ando's construction. However, our proof cannot proceed by constructing a series of $\star$ -isomorphisms, since $\mathcal{N}_F^\infty(T)$ is not the infinite composite of $\mathcal{N}_F(T)$.

Remark 5.7. Walker has also observed the relationship between the Coleman norm operator and Ando's criterion [Wal08, Chapter 5]. In particular, he has deduced that Ando's criterion is equivalent to (6) in [Wal08, Lemma 5.0.5 and (5.0.10)]. However, he did not prove Theorem 1.1 via the Coleman norm operator.

Remark 5.8. As mentioned in Remark 1.3, Zhu generalized Theorem 4.8 to apply to arbitrary complete local rings. To apply our proof to arbitrary complete local rings, we need to generalize the Coleman norm operator to such cases. However, recall in the argument around (10), we need R to be a domain so that we can count the number of roots of β , and we need $p \neq 0$ in R so that β is separable. Thus, there is a question whether the Coleman norm operator can be defined over arbitrary complete local rings.

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